Plotting Curves at Constant Speed

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Sometimes curves shall be plotted so that they represent a point moving along the curve at constant speed. Such curves may be simply relating one y coordinate to each x value (like $y = \sin(x)$) or they may be parametric curves which relate both, x and y, to a parameter p. Parametric curves may have multiple y values for the same x value. An example of such a curve is a circle. These curves are defined in parameter space as a function of a parameter p. Here p is an arbitrary parameter which may be linked to a physical parameter (like time, angle or distance), but this is not required. For each value of the parameter p there is one value for x and another value for y. Functions describe the relations $x(p)$ respectively $y(p)$.

Using Analytical Derivatives

We use the following parametric curve, which describes a circle:

$$x = \sin(p) \ , \tag{1}$$

$$y = \cos(p) \ . \tag{2}$$

The parameter p ranges from 0 to $2 \cdot \pi$ before the curve starts to repeat itself.

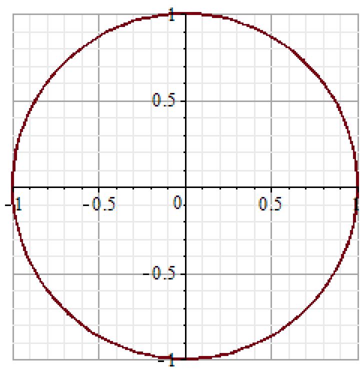

Figure 1: Plot of the parametric function x=sin(p) and y=cos(p).

If we want to plot a motion along such a curve we need the general expression for the velocity:

$$v = \frac{ds}{dt} \ . \tag{3}$$

An expression for the small distance ds determined from its horizontal and vertical components dx and dy was given by Pythagoras as

$$ds = \sqrt{dx^2 + dy^2} \ . \tag{4}$$

In order to find dx and dy as a function of p we differentiate the two equations describing the curve:

$$\frac{dx}{dp} = \cos(p) \text{ , and } \frac{dy}{dp} = -\sin(p) \text{ .} \tag{5}$$

Solving for dx and dy yields:

$$dx = \cos(p) \cdot dp \text{ , and } dy = -\sin(p) \cdot dp \text{ .} \tag{6}$$

Substituting dx and dy into the expression for the distance ds produces

$$ds = dp \cdot \sqrt{\cos(p)^2 + \sin(p)^2} \text{ .} \tag{7}$$

Finally we insert ds into the equation for the velocity so that we can find the required increment dp in the parameter space for a given velocity and time step dt. The time step dt can be chosen so that we obtain a nice resolution of the curve. It must then be kept constant during the plotting operation - we are not talking about Einstein and time warping here. With (7) the velocity becomes

$$v = \frac{dp \cdot \sqrt{\cos(p)^2 + \sin(p)^2}}{dt} \text{ .} \tag{8}$$

Noting that $\sin^2 + \cos^2 = 1$ we obtain the simple result

$$dp = v \cdot dt \text{ .} \tag{9}$$

For this example case we see that we could also use a constant angular step (in fact *p* is the circumferential angle). However, if we change the equation for y to $y = \cos\left(1/2 \cdot p\right)$ we will obtain uneven steps for dp.

General Approach

If we do not have analytical derivatives we can use the same approach but have to calculate the local derivatives by finite differences. This means we approximate for example dy/dp by the quotient $\Delta y/\Delta p = \left(y(p+\Delta p) - y(p-\Delta p)\right)/\left((p+\Delta p)-(p-\Delta p)\right)$. This equation is called a “central difference” because the gradient is determined from the symmetrical $\pm\Delta p$ variation of p around the center point p. It is more accurate than e.g. a “forward difference” which would evaluate the gradient by stepping only by $+\Delta p$ in the positive p direction. On the other hand the central difference requires two evaluations of the function $y(p)$ in addition to the current value, while the forward difference requires only one. The accuracy of the numerical approximation depends on the chosen step size Δp which should be “sufficiently” small. However, if Δp is too small, numerical errors in the differences and the division may become critical.

Using the numerical gradients we obtain for dx and dy:

$$dx = \frac{\Delta x}{\Delta p} \cdot dp \text{ , and } dy = \frac{\Delta y}{\Delta p} \cdot dp \text{ .} \tag{10}$$

And for the distance

$$ds = v \cdot dt = dp \cdot \sqrt{\left(\frac{\Delta x}{\Delta p}\right)^2 + \left(\frac{\Delta y}{\Delta p}\right)^2} \ . \tag{11}$$

The expression for dp can then be written in the general form

$$dp = \frac{v \cdot dt}{\sqrt{\left(\frac{\Delta x}{\Delta p}\right)^2 + \left(\frac{\Delta y}{\Delta p}\right)^2}} \ . \tag{12}$$

It describes the step size dp in the parameter space which corresponds to a time step dt with the constant speed v (which corresponds to the given distance $ds = v \cdot dt$).

Note 1

The algorithm can also be used to plot non-parametric functions of the form $y(x)$. In this case we simply replace the parameter $p := x$ in equation (12) and thus we obtain the following expression for the step size dx

$$dx = \frac{v \cdot dt}{\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}} \ . \tag{13}$$

Plotting the sine curve at constant speed we would obtain

$$dx = \frac{v \cdot dt}{\sqrt{1 + \cos(x)^2}} \ . \tag{14}$$

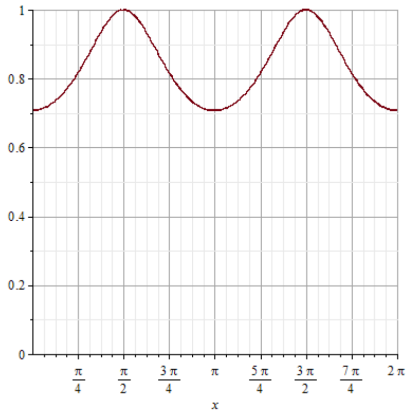

Figure 2: Required step size for plotting the sine curve at constant speed.

Note 2

If both derivatives dx/dp and dy/dp are zero at the same time, then no motion in x-y-space will occur when p is varied (see equation (11)). This situation corresponds to a singularity at point p and the basic plotting algorithm described below will hang. A similar problem occurs when the derivatives become infinite, i.e., one or both curves have a vertical asymptote.

One solution is to select an arbitrary new value close to the current p, say $p_2 = p + \Delta p$ and evaluate $x_2 = x\left(p_2\right)$ and $y_2 = y\left(p_2\right)$ at this new parameter value. Then the distance ds between $\left(x, y\right)$ and $\left(x_2, y_2\right)$ can be determined from (4) and a new value for the distance Δp can be selected until ds equals the prescribed value $v \cdot dt$. As the distance ds may be rather sensitive with respect to a variation of p close to a singular point, a relaxation technique may be necessary to update Δp in a monotonic converging manner.

Implementing a Plotting Algorithm

A simple plotting algorithm could use the following integration scheme (there are more accurate schemes available which are more complex and take more time):

1) We start at $t = 0$, $p = 0$, select a time step dt and a velocity v. Note that for constant velocity the distance travelled during each time step is also constant $ds = v \cdot dt$.
2) The coordinates from the two defining equations for x and y are calculated and we plot the point at $\left(x, y\right)$.
3) Next we determine the gradients dx/dp and dy/dp. We can use finite differences $\Delta x/\Delta p$ and $\Delta y/\Delta p$ if no analytical derivatives are available. We check the gradients for being at a singular point. If this is the case, the calculation of dp in step 4) must be changed accordingly.
4) Now we calculate dp from the equation (12) above (using the current value for p.
5) We can now advance to the next time step: we assign $t \leftarrow t + dt$ and $p \leftarrow p + dp$.
6) We repeat the cycle with step 2) until the desired end condition (e.g. a final time t_{end} or a final parameter p_{end}) has been reached.